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On the elastic Schwarzschild scattering cross section

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Abstract. We evaluate the differential cross section for scattering from a Schwarzschild source, the analogue of Rutherford scattering. The classical formula is compared with the Born approximation.

1. Introduction

The problem of a relativistic system characterized by a Schwarzschild metric has by now received exhaustive treatment (Darwin 1959, 1961, Zel'dovich and Novikov 1971) and the nature of the geodesics with their characteristic consequences are fairly well known. There is one aspect however, which, although qualitatively understood, has not to our knowledge been given any quantitative scrutiny, and this is the question of the differential scattering cross section from a Schwarzschild source, the direct analogue of Rutherford scattering. This paper is devoted to presenting the numerical results of such a computation. The main features which distinguish the Schwarzschild from the Rutherford problem are (i) the existence of a critical angular momentum below which capture occurs (absorption), (ii) multispiral scatterings which contribute to a given final scattering angle and divide the impact parameters into various 'zones', (iii) an infinite differential cross section in the backward direction which is, however, integrable and (iv) a profound difference between the classical cross section and the quantum mechanical Born approximation.

In § 2 we review the main properties of the geodesics (Synge 1960) and give implicitly the formula for the elastic differential cross section. In § 3 we give asymptotic expressions in the limit of large and critical angular momentum. The numerical results which interpolate between the two limits are presented in § 4 and are compared with the Born approximation results.

2. Geodesics and scattering

The Schwarzschild solution of Einstein's equations is described by the metric

$$\mathrm{d}s^2 = \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)-\left(1-\frac{2m}{r}\right)c^2\,\mathrm{d}t^2 \qquad (1)$$

where $m = GM/c^2$ and M is the mass of the scattering centre. Hereafter, we shall assume that r, θ, ϕ are indeed the polar coordinates which describe the motion of a particle in the gravitational field and do not require redefinition by further transformation of coordinates. If the motion of the test particle is taken to be in the $\theta = \pi/2$ plane, then the timelike geodesic equation (Synge 1960) is

$$\left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + 2\alpha^2 mu + \alpha^2(\beta^2 - 1) \equiv 2m(u-u_1)(u-u_2)(u-u_3) \quad (2)$$

where $u = 1/r$. The constants of motion α and β occurring above can be expressed in terms of the mass μ of the scattered particle, its energy E and its angular momentum l at asymptotic radial distances:

$$\alpha = \frac{\mu c}{l}, \qquad \beta = \frac{E}{\mu c^2}. \quad (3)$$

It is sufficient for our purposes to consider the case when the roots $u_1 \leqslant u_2 \leqslant u_3$ are all real. We are interested in situations where the test particle is initially at an infinite distance from the centre, in which case two possibilities arise: (i) scattering of the particle, occurring when $u_1 \leqslant 0 < u_2 < u_3$; (ii) capture of the particle, when $u_1 \leqslant 0 < u_2 = u_3$. (Capture also takes place when u_1 is negative and u_2 and u_3 are complex conjugates with positive real part.)

Integration of equation (2) gives the trajectory equation

$$u = u_1 + (u_2 - u_1)\,\mathrm{sn}^2\,[K - \tfrac{1}{2}\phi\{2m(u_3 - u_1)\}^{1/2}] \quad (4)$$

where we have chosen $\phi = 0$ to correspond to the perihelion ($u = u_2$) so that the constant of integration is

$$K = \int_0^1 dy\{(1-y^2)(1-k^2y^2)\}^{-1/2}$$

with (5)

$$k^2 = \frac{u_2 - u_1}{u_3 - u_1}.$$

In fact $4K$ is the period of the elliptic function sn and k is its modulus. The total 'angle of deviation' χ of the test particle (see figure 1) is

$$\chi = 2\phi(u = 0) - \pi \quad (6)$$

which is an implicit equation relying on the solution of equation (4).

It is worthwhile to follow the trajectories as we decrease the angular momentum of the system (ie as the impact parameter is reduced) in order to understand the nature of the scattering and thereby to compute the cross section (see figure 1). For small α (large l) there is very little deviation of the trajectory from a straight line and the perihelion is far from the scattering centre (small u_2). As α is increased so is the deviation until a stage is reached when the particle is returned to its initial line of motion; further increase of α results in multispiral motion about the centre (one, two, ... loops), until finally when α approaches the critical values α_c the particle is no longer able to escape to infinity but spirals into the centre. This is one of the features which makes numerical computations that much more difficult than in the Rutherford case.

To arrive at a formula for the differential scattering cross section we assume a steady state situation with a stream of parallel moving particles which are continuously moving

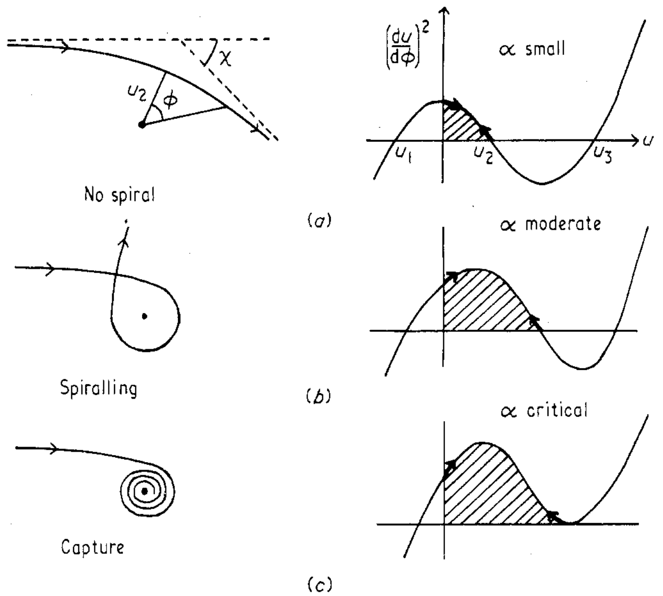

Figure 1. Trajectories and cubics in the corresponding geodesic equations for various values of α.

in from infinity, following trajectories determined by their respective impact parameters

$$b = \frac{l}{p} = \alpha^{-1}(\beta^2 - 1)^{-1/2},$$

and are scattered into the detecting apparatus. We note first of all that the measured scattering angle is $\theta = |\chi - 2n\pi|$ where n is chosen so that $0 \leqslant \theta \leqslant \pi$. The detectors collect particles which have undergone no loops, 1 loop, 2 loops, ... —in this way the impact parameters can be divided into 'Fresnel-like zones'—and theoretically we shall have to sum over all these possibilities. If there were no spiralling we would obtain the conventional formula,

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{1}{\alpha^3(\beta^2-1)\sin\theta}\left|\frac{d\alpha}{d\theta}\right|.$$

This has now to be replaced by

$$\frac{d\sigma}{d\Omega} = \frac{1}{\beta^2 - 1}\sum_n \frac{1}{\alpha_n^3}\left|\frac{d\alpha_n}{d\cos\theta}\right| \tag{7}$$

where α_n connotes the range of α values which result in an n spiral scattering having $\theta = \pm(\chi - 2n\pi)$. In this steady state approach the delays between particles arriving at the detectors having undergone different numbers of spirals are irrelevant.

3. Asymptotic results

We are able to obtain asymptotic expansions about two limits, $\alpha = 0$ and $\alpha = \alpha_c$. These serve to provide useful checks on numerical calculations (given in the next section) which interpolate between the two values.

By expressing u_i in terms of α, we see that small α corresponds to small k of the elliptic function since

$$k^2 = 4m\alpha(\beta^2-1)^{1/2}\{1-2m\alpha(\beta^2-1)^{1/2}+\ldots\}. \tag{8}$$

Let us therefore expand both sides of our basic formula

$$u_1(u_1-u_2)^{-1} = \mathrm{sn}^2[K-\tfrac{1}{4}(\pi+\chi)\{2m(u_3-u_1)\}^{1/2}] \tag{4}$$

in powers of k^2. This involves expansions for the roots u_i of the original cubic equation (2)

$$\begin{aligned} u_1 &= -\frac{k^2}{4m}\left(1+\frac{k^2(\beta^2-2)}{4(\beta^2-1)}+\cdots\right) \\ u_2 &= \frac{k^2}{4m}\left(1+\frac{k^2(3\beta^2-2)}{4(\beta^2-1)}+\cdots\right) \\ u_3 &= \frac{1}{2m}\left(1-\frac{k^4\beta^2}{4(\beta^2-1)}+\cdots\right). \end{aligned} \tag{9}$$

The period of the elliptic function (Magnus *et al* 1966) can likewise be expanded,

$$K = \frac{\pi}{2}\left(1+\frac{k^2}{4}+\frac{9k^4}{64}+\cdots\right)$$

and inserted in the Jacobi elliptic function expansion

$$\begin{aligned} \mathrm{sn}\,\xi = &\left(1+\frac{k^2}{16}+\frac{7k^4}{256}+\cdots\right)\sin\left(\frac{\pi\xi}{2K}\right)+\left(\frac{k^2}{16}+\frac{k^4}{32}+\cdots\right)\sin\left(\frac{3\pi\xi}{2K}\right) \\ &+\left(\frac{k^4}{256}+\cdots\right)\sin\left(\frac{5\pi\xi}{2K}\right)+\cdots \end{aligned} \tag{10}$$

to give

$$\chi = \frac{k^2(2\beta^2-1)}{2(\beta^2-1)}\left\{1+\left(1+\frac{3\pi(5\beta^2-1)}{16(2\beta^2-1)}\right)\frac{k^2}{2}+\cdots\right\}$$

or

$$\theta = 2m\alpha\left(\frac{2\beta^2-1}{(\beta^2-1)^{1/2}}+\frac{3\pi}{8}m\alpha(5\beta^2-1)+\cdots\right). \tag{11}$$

Substituting in equation (7) we obtain the *small angle* differential cross section,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{2m(2\beta^2-1)}{\theta^2(\beta^2-1)}\right)^2+\frac{3\pi m^2(5\beta^2-1)}{4\theta^3(\beta^2-1)}+\ldots. \tag{12}$$

We see that the leading behaviour agrees with the small angle Rutherford formula in the nonrelativistic limit

$$\beta^2-1 = \frac{2T}{\mu c^2} \ll 1$$

(where T corresponds to the asymptotic kinetic energy) namely

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{GM\mu}{4T}\right)^2\frac{1}{\sin^4\frac{1}{2}\theta} \tag{13}$$

as $\theta \to 0$. However, observe that discrepancies occur in next to leading order.

Near the other limit $\alpha = \alpha_c$ there is a near equality of the roots u_2 and u_3. We therefore expand about the critical value

$$\alpha_c = \left(\frac{\beta(9\beta^2-8)^{3/2}-27\beta^4+36\beta^2-8}{32m^2}\right)^{1/2} \tag{14}$$

where

$$\begin{aligned} u_{1c} &= \frac{1-2(1-12m^2\alpha_c^2)^{1/2}}{6m} \\ u_{2c} = u_{3c} &= \frac{1+(1-12m^2\alpha_c^2)^{1/2}}{6m} \end{aligned} \tag{15}$$

and $k^2 = 1$. Thus a suitable expansion parameter is $1-k^2$ in terms of which we may write (Magnus *et al* 1966)

$$\begin{aligned} u_1 &= u_{1c}+\frac{u_{2c}-u_{1c}}{4}\frac{\beta^2-1+2mu_{1c}}{\beta^2-1+2mu_{2c}}(1-k^2)^2+\dots \\ u_2 &= u_{2c}-\tfrac{1}{2}(u_{2c}-u_{1c})(1-k^2)+\dots \\ u_3 &= u_{3c}+\tfrac{1}{2}(u_{2c}-u_{1c})(1-k^2)+\dots \\ \alpha &= \alpha_c-\frac{m(u_{2c}-u_{1c})^3(1-k^2)^2}{4\alpha_c(\beta^2-1+2mu_{2c})}+\dots \end{aligned} \tag{16}$$

$$\begin{aligned} K &= \sum_{n=0}^{\infty}\frac{(1/2)n(1/2)n}{(n!)^2}\{\psi(n+1)-\psi(n+\tfrac{1}{2})-\tfrac{1}{2}\ \ln(1-k^2)\}(1-k^2)^n \\ &= -\tfrac{1}{2}\ln(1-k^2)\{1+\tfrac{1}{4}(1-k^2)+\dots\}+2\ln 2+\tfrac{1}{4}(2\ln 2-1)(1-k^2)+\dots . \end{aligned} \tag{17}$$

Also from the exact property of the Jacobi elliptic function,

$$\mathrm{sn}(x,k) = \frac{\mathrm{i}\,\mathrm{sn}(-\mathrm{i}x,(1-k^2)^{1/2})}{\mathrm{cn}\,(-\mathrm{i}x,(1-k^2)^{1/2})}$$

we are able to use again expansions of elliptic functions of small moduli to deduce that

$$\mathrm{sn}(x,k) = \tanh x+\tfrac{1}{4}(\tanh x-x\,\mathrm{sech}^2\,x)(1-k^2)+\dots . \tag{18}$$

Equation (4) then reduces to

$$\left(\frac{u_{1c}}{u_{1c}-u_{2c}}\right)^{1/2} = \tanh[2\ln 2-\tfrac{1}{2}\ln(1-k^2)-\tfrac{1}{4}(\chi+\pi)\{2m(u_{2c}-u_{1c})\}^{1/2}]+\mathrm{O}(1-k^2)$$

giving

$$\begin{aligned} \chi &= -\pi+\frac{4}{\{2m(u_{2c}-u_{1c})\}^{1/2}}\left\{-\tfrac{1}{2}\ln(1-k^2)+2\ln 2-\tanh^{-1}\left(\frac{u_{1c}}{u_{1c}-u_{2c}}\right)^{1/2}\right\}+\mathrm{O}(1-k^2) \\ &= -\pi+\frac{1}{\{2m(u_{2c}-u_{1c})\}^{1/2}}\left\{-\ln(\alpha_c-\alpha)+8\ln 2-4\tanh^{-1}\left(\frac{u_{1c}}{u_{1c}-u_{2c}}\right)^{1/2}\right. \\ &\qquad \left.-\ln\frac{4\alpha_c(\beta^2-1+2mu_{2c})}{m(u_{2c}-u_{1c})^3}\right\}+\mathrm{O}((\alpha_c-\alpha_n)^{1/2}) \end{aligned} \tag{19}$$

from which we must subtract off the appropriate number of $2n\pi$ to relate to the scattering angle θ. Thus as $\alpha \to \alpha_c$ the term $\ln(\alpha_c - \alpha)$ dominates and controls the number of spirals. It follows that

$$\frac{d\chi}{d\alpha_n} = \frac{1}{\{2m(u_{2c}-u_{1c})\}^{1/2}(\alpha_c-\alpha_n)} + \ldots = \frac{\exp[\{2m(u_{2c}-u_{1c})\}^{1/2}(2n\pi \pm \theta)]}{\{2m(u_{2c}-u_{1c})\}^{1/2}} + \ldots \tag{20}$$

and therefore the ratio of successive contributions to the sum in equation (7) is essentially given by

$$\frac{\alpha_n^3}{\alpha_{n+1}^3}\exp[-\pi\{2m(u_{2c}-u_{1c})\}^{1/2}].$$

The relationship between the roots of the cubic, and the physical requirement $\beta^2 \geqslant 1$ mean that $2m(u_{2c}-u_{1c})^{1/2} \geqslant \frac{1}{2}$ and therefore the successive 'partial' differential cross sections decrease as $\exp(-\pi/\sqrt{2}) \simeq 0{\cdot}1$. Therefore the sum over spirals converges rapidly for large n and one can write

$$\frac{d\sigma}{d\Omega} = \text{zero spiral cross section} + \frac{\{2m(u_{2c}-u_{1c})\}^{1/2}}{(\beta^2-1)\sin\theta}$$
$$\times \sum_{n=1}^{\infty} \frac{\exp[-\{2m(u_{2c}-u_{1c})\}^{1/2}(2n\pi+\theta)] + \exp[-\{2m(u_{2c}-u_{1c})\}^{1/2}(2n\pi-\theta)]}{\alpha_n^3}$$
$$= \text{zero spiral cross section}$$
$$+ \frac{2\{2m(u_{2c}-u_{1c})\}^{1/2}}{(\beta^2-1)\alpha_c^3\sin\theta}\exp[-2\pi\{2m(u_{2c}-u_{1c})\}^{1/2}]\cosh[\{2m(u_{2c}-u_{1c})\}^{1/2}\theta] + \ldots . \tag{21}$$

4. Intermediate results

For general values of angular momentum lying between the two limiting cases considered above the differential cross section has to be evaluated numerically. In order to do this we adopt the integrated form of equation (4),

$$\chi = -\pi + 2\int_0^{u_2} du\{2m(u-u_1)(u-u_2)(u-u_3)\}^{-1/2}.$$

Because the integrand has a singularity at the upper limit it is convenient to remove this by partial integration

$$\chi = -\pi + 4\left(\frac{-u_2}{2mu_1u_3}\right)^{1/2} - 2\int_0^{u_2}\left(\frac{u_2-u}{2m}\right)^{1/2}\frac{(u_1+u_3-2u)\,du}{\{(u-u_1)(u_3-u)\}^{3/2}}. \tag{22}$$

Now the integral can be evaluated by computer and in figure 2 we have plotted the differential cross section against momentum transfer t (as is more conventional in high energy physics),

$$\frac{d\sigma}{dt} = \frac{\pi}{\mu^2c^2(\beta^2-1)}\frac{d\sigma}{d\Omega}.$$

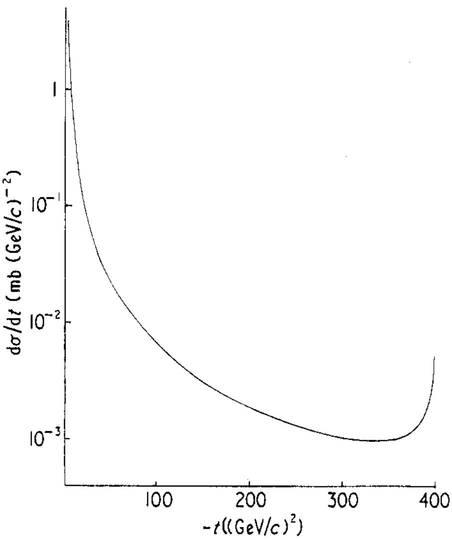

Figure 2. Differential cross section for a particle of mass 1 GeV/c^2 and momentum 10 GeV/c, with a scattering centre of mass 10^{-15} cm in gravitational units.

In our calculation we have given the test particle a typical mass of 1 GeV/c^2 and momentum 10 GeV/c; the mass of the scattering centre has been taken as 10^{-15} cm in gravitational units. The only point of note is the appearance of an infinite cross section in the backward direction which is, however, integrable. The Rutherford cross section, given by equation (13), remains finite in this direction because $db/d\chi$ vanishes there. In our calculation this does not occur since the scattering angle increases indefinitely as b decreases. The infinity in the forward direction is of course the normal one of Rutherford scattering and can be treated in the usual way.

For Coulomb scattering it is a well known fact that in a quantum mechanical treatment the Born approximation reproduces exactly the Rutherford answer and that the higher order contributions do not affect the result apart from endowing the scattering amplitude with an overall phase. It is therefore of interest to determine the Born approximation in the Schwarzschild analogue and compare it with the classical cross section just determined. Now the Green function for a scalar test particle moving in a Schwarzschild metric,

$$(\partial^2+\mu^2)\phi+\partial_\nu\{(g^{\mu\nu}-\eta^{\mu\nu})\partial_\mu\phi\} = 0$$

satisfies the equation

$$\Delta'(p) = (p^2-m^2)^{-1}\left(1+\int\frac{d^4p'}{(2\pi)^4}\,d^4x\,\exp\{i(\boldsymbol{p}-\boldsymbol{p}')\,.\,\boldsymbol{x}\}p'_\mu p_\nu(g^{\mu\nu}(x)-\eta^{\mu\nu})\Delta'(p')\right).$$

Thus the Born approximation to the scattering amplitude in the Schwarzschild geometry† reduces to

$$T(\boldsymbol{p}',\boldsymbol{p}) = \int d^3\boldsymbol{x}\,\exp\{-i(\boldsymbol{p}-\boldsymbol{p}')\,.\,\boldsymbol{x}\}\left\{\frac{2m(\boldsymbol{p}^2+\mu^2)}{r-2m}+\frac{2m}{r}\left(\frac{\boldsymbol{x}\,.\,\boldsymbol{p}\boldsymbol{x}\,.\,\boldsymbol{p}'}{r^2}\right)\right\}$$

† Dr J Strathdee has evaluated a similar expression for the Born amplitude using isotropic coordinates and we are indebted to him for acquainting us with the method and results.

or

$$T(k) = 2m\pi\left(1+\frac{4E^2}{k^2}-\frac{8mE^2}{k}\{\sin(2mk)\,\mathrm{ci}(-2mk)+\cos(2mk)\,\mathrm{si}(-2mk)\}\right)$$

where $\boldsymbol{k}$ = momentum transfer, E = (conserved) energy of test particle. The corresponding cross section is depicted in figure 3 and we see that it differs considerably

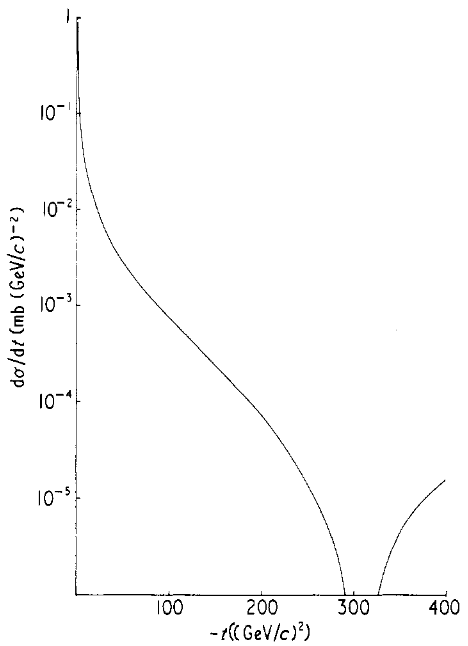

Figure 3. Differential cross section in the Born approximation for a particle of mass 1 GeV/c^2 and momentum 10 GeV/c, with a scattering centre of mass 10^{-15} cm in gravitational units.

from the classical Schwarzschild cross section. It would be rather optimistic although not inconceivable to expect that the exact Green function can reproduce the classical result. However, there are good reasons for doubting such simple minded generalizations to the quantum mechanical situation. In the first place in any realistic scattering process we should be considering the motion of a wave packet, specified by some typical dispersion and size, as it approaches the scattering centre; thus we may imagine a plane-fronted wave of large enough dimensions and characterized by appropriate width approaching the potential singularity. In the Rutherford problem one discovers a recognizable outcoming quasiradial packet emerging after some time, as one can simply ascertain by following the trajectories of a cloud of classical particles which simulate the wave packet. On the other hand, in the Schwarzschild problem this is not what happens; as a result of delays occasioned by multispiralling (due to the dispersion in the packet and the effects of near critical impact parameters) one would expect to get successions of progressively weaker outcoming radial waves, so much so that it is difficult to invisage how a viable radial wave packet can emerge and be interpreted as a particle. It may be that this phenomenon is connected with the apparent singular behaviour of the radial wavefunction at the Schwarzschild radius which suggests a redefinition of the radius r and time t to other more appropriate coordinates (Eddington 1924, Kruskal 1960).

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